International Applied Mechanics, Vol. 31, No. 10, 1995

MINIMALLY PARAMETRIC FORM OF THE EQUATIONS OF MOTION OF AIRCRAFT

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UDC 531.01 + 629.7.015

1. INTRODUCTION

The application of mathematical models of the motion of aircraft, not only for their design but also directly in active control systems with flight prediction, has renewed practical interest in the problem of how to represent the mathematical models in the most economical (from the computational point of view) form. A problem of vital interest in this regard is the representation of a model in "dimensionless" form or, more precisely, the reduction of the mathematical model to such a ("canonical") form with minimization of the number of physical parameters. The conventional techniques of mechanics based on dimensional considerations and the π-theorem are inadequate for solving this problem. Their inadequacy lies in the fact that dimensionless variables can also participate effectively in the transformations used in flight mechanics [4, 7], but such transformations are known only for certain special problems of flight dynamics and are based on more or less lucky guesses. It is understandable, then, that they cannot be used in treating other models of aircraft motion. On the other hand, modern group-analytic techniques for the investigation of differential equations represent a unified methodological foundation for treating the problem in a general context. Consequently, the present study has a twofold objective: on the one hand, to reduce the equations of the mathematical model of longitudinal motion of an aircraft to a canonical form and, on the other, to demonstrate the group approach and disclose algorithmic features of the solution of such problems.

2. GROUP EQUIVALENCE: CONCEPT OF ESSENTIAL PARAMETERS

The solution of the problem of reducing the mathematical model to canonical form involves the analysis of special groups of continuous transformations, viz.: groups of equivalence transformations.

Definition 1 ([5] and [1], p. 18). A parameter is said to be essential if the system of equations does not admit a continuous group of equivalence transformations with respect to it.

Other required definitions can be found in [5]. We note that the basic property of an equivalence group is the fact that it takes the solutions of the given system corresponding to certain values of the parameters into solutions of that system, but now corresponding to different values of the parameters. The required changes of variables and parameters are determined as invariants of the corresponding groups of equivalence transformations. Based on the well-known relationship between groups and Lie algebras, we carry out our analysis at the infinitesimal level, working with group generators.

3. REDUCTION OF THE MATHEMATICAL MODEL OF LONGITUDINAL AIRCRAFT MOTION TO CANONICAL FORM

We consider a mathematical model of longitudinal motion of an aircraft as represented by a system of differential equations of the form

Air Force Institute, Kiev. Translated from Prikladnaya Mekhanika, Vol. 31, No. 10, pp. 81-87, October, 1995. Original article submitted April 18, 1994.

1063-7095/95/3110-0852$12.50

$$\frac{dh}{dt} = V\sin\theta, \quad \frac{dL}{dt} = V\cos\theta, \tag{1}$$

$$\frac{dV}{dt} = \frac{1}{m}\left(P - (Ac^2 + B)\frac{\rho V^2}{2}S - mg\sin\theta\right), \quad \frac{d\theta}{dt} = \frac{1}{mV}\left(c_y\frac{\rho V^2}{2}S - m\cos\theta\right), \tag{2}$$

where the height h, the range L, the velocity V, the angle of inclination θ of the trajectory, and the time t are phase variables; the thrust factor P and the lift coefficient c_y are control variables; the mass m, the characteristic area S, the air density ρ, the free-fall acceleration g, and the polar aerodynamic characteristics (A, B) are constant coefficients.

In the first stage we classify the control variables P and c_y as parameters and seek the infinitesimal group operator in the form

$$\begin{gathered} X = \xi^0\partial_t + \xi^1\partial_h + \xi^2\partial_L + \xi^3\partial_V + \xi^4\partial_\theta + \\ + \varphi^1\partial_P + \varphi^2\partial_m + \varphi^3\partial_g + \varphi^4\partial_S + \varphi^5\partial_\rho + \varphi^6\partial_{c_y} + \varphi^7\partial_A + \varphi^8\partial_B; \\ \xi^i = \xi^i(t,h,L,V,\theta), \quad \varphi^j = \varphi^j(P,m,\rho,S,c_y,g,A,B), \\ i = 0,1,\dots,4; \quad j = 1,2,\dots,8; \quad \partial_k \equiv \frac{\partial}{\partial_k}. \end{gathered} \tag{3}$$

We associate with the system (1), (2) the differential operator

$$\begin{gathered} X_0 = \partial_t + \frac{1}{m}\left(P - (Ac_y^2 + B)\frac{\rho V^2}{2}S - mg\sin\theta\right)\partial_V + \\ + \frac{1}{mV}\left(c_y\frac{\rho V^2}{2}S - mg\cos\theta\right)\partial_\theta + V\sin\theta\,\partial_h + V\cos\theta\,\partial_L, \end{gathered} \tag{4}$$

which is the operator of total differentiation with respect to time by virtue of the system (1), (2), and we write the invariance condition in the form

$$[X_0, X] = \lambda X_0, \tag{5}$$

where [.,.] denotes the operator commutator. The search for the coefficients of the operator (3) is constructive in that the coefficients ξ^i and φ^j depend on different sets of variables, thereby affording additional possibilities for decoupling the governing system after the expansion of relation (5). We note that some operators of the form (3) can be found through the well-known π-theorem rather than on the basis of relation (5). In particular, the invariance of the system (1), (2) under scale transformations of time, mass, and length is characterized by the respective operators

$$X_1 = t\partial_t - V\partial_V - 2P\partial_P - 2g\partial_g, \tag{6}$$

$$X_2 = h\partial_h + P\partial_P + \rho\partial_\rho, \tag{7}$$

$$X_3 = h\partial_h + L\partial_L + V\partial_V + P\partial_P + 2S\partial_S + g\partial_g - 3\rho\partial_\rho. \tag{8}$$

Moreover, it is readily noted that the parameters ρ and S are encountered only in the combination ρS, i.e., the operator

$$X_4 = \rho\partial_\rho - S\partial_S \tag{9}$$

is admissible. The four operators (6)-(9) form a complete system in nine variables (t, h, L, V, m, ρ, S, g, P) and, accordingly, have five functionally independent solutions, which are conveniently written in the form

$$\hat{t} = t\left(\frac{\rho g S}{2m}\right)^{1/2}, \quad \hat{V} = V\left(\frac{\rho S}{2mg}\right)^{1/2}, \quad \hat{P} = \frac{P}{mg}, \qquad \hat{h} = h\frac{\rho S}{2m}, \quad \hat{L} = L\frac{\rho S}{2m}. \tag{10}$$

In the new variables the system (1), (2) assumes the form (here we drop the diacritical ^)

$$\dot{h} = V\sin\theta, \quad \dot{L} = V\cos\theta, \tag{11}$$

$$\dot{V} = P - (Ac_y^2 + B)V^2 - \sin\theta, \quad \dot{\theta} = \frac{1}{V}(c_y V^2 - \cos\theta). \tag{12}$$

Since the right-hand sides do not depend on h or L, the latter expression admits with respect to these variables the translation group with generators

$$X_5 = \partial_h, \quad X_6 = \partial_L. \tag{13}$$

It is therefore convenient to continue the analysis of the "essentialness" of the remaining parameters (P, A, B, c_y) in application to the reduced model, i.e., to consider Eqs. (12) with the operator

$$X_0 = \partial_t + (P - (Ac_y^2 + B)V^2 - \sin\theta)\partial_V + \frac{1}{V}(c_y V^2 - \cos\theta)\partial_\theta, \tag{4a}$$

and to seek a symmetry operator in the form

$$X = \tau(t,V,\theta)\partial_t + \xi(t,V,\theta)\partial_V + \eta(t,V,\theta)\partial_\theta + \varphi^1\partial_P + \varphi^2\partial_{c_y} + \varphi^3\partial_A + \varphi^4\partial_B, \quad \varphi^i = \varphi^i(A,B,c_y,P). \tag{14}$$

The invariance condition has the same form as before, (5), and leads to governing equations of the form

$$\begin{aligned}\xi_t &+ (P - (Ac_y^2 + B)V^2 - \sin\theta)\xi_V + \frac{1}{V}(c_y V^2 - \cos\theta)\xi_\theta - \varphi^1 + \\ &+ 2Ac_y V^2\varphi^2 + c_y^2 V^2\varphi^3 + V^2\varphi^4 + \eta\cos\theta = (\tau_t + (P - (Ac_y^2 + B)V^2 - \\ &- \sin\theta)\tau_V + \frac{1}{V}(c_y V^2 - \cos\theta)\tau_\theta)(P - (Ac_y^2 + B)V^2 - \sin\theta);\end{aligned} \tag{15}$$

$$\begin{aligned}\eta_t &+ (P - (Ac_y^2 + B)V^2 - \sin\theta)\eta_V + \frac{1}{V}(c_y V^2 - \cos\theta)\eta_\theta - \\ &- \xi\left(c_y + \frac{\cos\theta}{V^2}\right) - \eta\frac{\sin\theta}{V} - V\varphi^2 = (\tau_t + (P - (Ac_y^2 + B)V^2 - \sin\theta)\tau_V + \\ &+ \frac{1}{V}(c_y V^2 - \cos\theta)\tau_\theta)\frac{1}{V}(c_y V^2 - \cos\theta).\end{aligned} \tag{16}$$

We begin the analysis of the system (15), (16) with Eq. (16). We make use of the fact that (16) involves only one coefficient: φ^2. Grouping terms in like powers of c_y, we observe that (16) represents the sum of a cubic (in c_y) polynomial and the term $V\varphi^2$, where (by assumption) $\varphi^2 = \varphi^2(A, B, c_y, P)$. Consequently, if (16) is differentiated four times with respect to c_ym, we obtain

$$\frac{\partial^4\varphi^2}{\partial c_y^4} = 0, \tag{17}$$

so that

$$\varphi^2 = C_1 c_y^3 + C_2 c_y^2 + C_3 c_y + C_4, \tag{18}$$

where $C_i = C_i(A, B, P)$.

Substituting Eq. (18) into (16) and decoupling with respect to powers of c_y, we obtain the governing equations

$$C_1 = A\tau_V V^2, \tag{19}$$

$$A\eta_V + C_2 = -V\tau_\theta, \tag{20}$$

$$V\eta_\theta - \xi - VC_3 = \tau_V V + V(P - BV^2 - \sin\theta)\tau_V, \tag{21}$$

$$\eta_t + V(P - BV^2 - \sin\theta)\eta_V - \frac{\cos\theta}{V}\eta_\theta - \xi\frac{\cos\theta}{V^2} - \eta\frac{\sin\theta}{V} - VC_4 =$$
$$= \left(\tau_t + (P - BV^2 - \sin\theta)\tau_V - \frac{\cos\theta}{V}\tau_\theta\right)\left(-\frac{\cos\theta}{V}\right). \tag{22}$$

Differentiating Eq. (19) with respect to A, we obtain

$$\frac{\partial C_1}{\partial A} = \tau_V V^2 = k_1, \tag{23}$$

where $k_1 = \text{const}$.

Relation (23) must be satisfied, because τ_V and C_1 depend on different sets of variables. Integrating (23), we have

$$C_1 = k_1 A + \hat{C}_1(B, P), \tag{24}$$

$$\tau = \frac{k_1}{V} + \tau^1(\theta, t). \tag{25}$$

The substitution of Eqs. (24) and (25) into (19) yields the relation $\hat{C}_1(B, P) = 0$. Equation (20) is similarly analyzed; differentiating with respect to A, we have

$$\eta_V = \frac{\partial C_2}{\partial A} = k_2. \tag{26}$$

The integration of (26) gives

$$\eta = k_2 V + \eta^1(t, \theta), \tag{27}$$

$$C_2 = -k_2 A + \hat{C}_2(B, P). \tag{28}$$

The substitution of Eqs. (27) and (28) into (20) yields

$$\hat{C}_2(B, P) = -V\tau^1_\theta. \tag{29}$$

Decoupling (29) with respect to powers of V, we infer that $\hat{C}_1(B, P) = 0$ and

$$\tau^1_\theta = 0, \tag{30}$$

i.e., $\tau^1 = \tau^1(t)$. We are now ready to investigate Eq. (21); we begin by substituting the values of τ_V and η_θ from Eqs. (25) and (27), whereupon ξ is given by the expression

$$\xi = V(\eta^1_\theta - C_3 - \tau^1_t) + \frac{k_1(P - BV^2 - \sin\theta)}{V}, \tag{31}$$

and we substitute Eq. (31) into (22). In the resulting equation

$$\eta^1_t - k_2(P - BV^2 - \sin\theta) - \frac{\cos\theta}{V}\eta^1_\theta - (\eta^1_\theta - C_3 - \tau^1_t)\frac{\cos\theta}{V} -$$
$$- (\eta^1 - k_2 V)\frac{\sin\theta}{V} - VC_4 = -\frac{\cos\theta}{V}\tau^1_t \tag{32}$$

we decouple the variables in powers of V:

$$V^2:\ \ k_2=0, \tag{33}$$

$$V:\ \ C_4=0, \tag{34}$$

$$V^0:\ \ \eta^1_t=0,\ \text{i.e.,}\ \ \eta^1=\eta^1(\theta), \tag{35}$$

$$V^{-1}:\ \ 2\cos\theta(\tau^1_t-\eta^1_\theta)+C_3\cos\theta-\eta^1\sin\theta=0. \tag{36}$$

Dividing Eq. (36) by $-2\cos\theta\neq 0$, we note that the left-hand side becomes a function of θ only, and the right-hand side becomes a function of t only, so that the equality can hold only in the case

$$\tau^1_t=\eta^1_\theta+\frac{1}{2}\eta^1\operatorname{tg}\theta-\frac{C_3}{2}=k_3. \tag{37}$$

Integrating (37), we have

$$\tau^1=k_3t+k_4, \tag{38}$$

$$\eta^1=\sqrt{2}\left(k_3+\frac{C_3}{2}\right)G(\theta,1/\sqrt{2})\cos^{1/2}\theta+k_5\cos^{1/2}\theta, \tag{39}$$

where

$$G(\varphi,k)=\int\limits_0^\varphi(1-k^2\sin^2\varphi)^{-1/2}d\varphi$$

is an elliptic integral of the first kind. We conclude the analysis by substituting the values obtained for η, ξ, τ, and φ^2 into Eq. (15) to determine φ^1, φ^3, and φ^4. We obtain the equation

$$\begin{aligned}&(P-(Ac_y^2+B)V^2-\sin\theta)(\eta^1_\theta-C_3-k_3)+(c_yV^2-\cos\theta)\eta^1_{\theta\theta}-\\&-\varphi^1+2V^2(Ac_y^2+B)(\eta^1_\theta-C_3-k_3)+2Ac_yV^2(k_1Ac_y^3+C_3c_y)+\\&+c_y^2V^2\varphi^3+V^2\varphi^4+\eta\cos\theta=\left[k_3+[P-(Ac_y^2+B)V^2-\sin\theta]\frac{k_1}{V^2}\right]*\\&*[P-(Ac_y^2+B)V^2-\sin\theta],\end{aligned} \tag{40}$$

which can be satisfied identically in V and θ. Decoupling with respect to V, we have

$$V^{-2}:\ \ k_1=0, \tag{41}$$

$$V^0:\ \ (P-\sin\theta)(\eta^1_\theta-C_3-2k_3)-\eta^1_{\theta\theta}\cos\theta-\varphi^1+\eta\cos\theta=0, \tag{42}$$

$$V^2:\ \ -(Ac_y^2+B)(\eta^1_\theta-C_3-2k_3)+c_y\eta^1_{\theta\theta}+2AC_3c_y^2+c_y^2\varphi^3+\varphi^4=0. \tag{43}$$

It follows from (42) that the equality holds only under the condition

$$\varphi^1=0, \tag{44}$$

$$C_3 = -2k_3, \tag{45}$$

$$\eta^1 = 0, \text{ i.e., } k_5 = 0. \tag{46}$$

Equation (43) is then transformed to

$$\varphi^4 + \varphi^3 c_y^2 + 2k_3(B - Ac_y^2) = 0. \tag{47}$$

Solving Eq. (47) for φ^4 and calculating the final values of the coefficients, we have

$$\tau = k_3 t + k_4, \quad \xi = k_3 V, \quad \eta = 0, \quad \varphi^1 = 0,$$
$$\varphi^2 = -2k_3 c_y, \quad \varphi^4 = -\varphi^3 c_y^2 - 2k_3(B - Ac_y^2). \tag{48}$$

Taking expressions (48) into account, we can choose the basis of the Lie algebra of the group of equivalence transformations of the system (11), (12) in the form

$$X_7 = \partial_t; \quad X_8 = \partial_A - c_y^2 \partial_B; \quad X_9 = t\partial_t + V\partial_V - 2c_y \partial_{c_y} + 2A\partial_A - 2B\partial_B. \tag{49}$$

The operators $X_1, \ldots, X_9$ thus form the nine-dimensional (maximum) Lie algebra of the equivalence group of the system (1), (2). The operators X_5, X_6, and X_7 form the kernel of the algebra [i.e., are admitted by the system (1), (2) for any values of the parameters] and signify the invariance of the system (1), (2) under translations in time (X_7) and space (X_5 and X_6). The operators X_2, X_4, and X_8, which act only in parameter space, signify that some of the parameters are nonessential and, accordingly, there is no need to detail the system (1), (2) in analyzing the proper motions. In view of the above-determined operators (49) the formulas for the transformations of variables now have the form

$$k = \frac{c_y}{Ac_y^2 + B}; \quad \hat{V} = V\left(\frac{\rho S(Ac_y^2 + B)}{2mg}\right)^{1/2}, \quad \hat{P} = \frac{P}{mg};$$
$$\hat{t} = t\left(\frac{\rho g S(Ac_y^2 + B)}{2m}\right)^{1/2}, \quad \hat{h} = h\frac{\rho S}{2m}; \quad \hat{L} = L\frac{\rho S}{2m}. \tag{50}$$

On the basis of (50) the system (1), (2) assumes the form (we drop the diacritical ^)

$$\dot{h} = V\sin\theta, \quad \dot{L} = V\cos\theta, \tag{51}$$

$$\dot{V} = P - V^2 - \sin\theta, \quad \dot{\theta} = \frac{1}{V}(kV^2 - \cos\theta) \tag{52}$$

and contains only two essential parameters: the thrust–weight ratio P and the lift–drag ratio k. Under the stated assumptions, therefore, these two parameters are the ones that determine the qualitative behavior (proper motions) of the aircraft in the vertical plane.

Moreover, the presence of the operators X_5, X_6, and X_7 enables us to reduce the analytical investigation of the system (51), (52) to the equation

$$\frac{dV}{d\theta} = V\frac{P - V^2 - \sin\theta}{kV^2 - \cos\theta}, \tag{53}$$

which the substitution $V^2 = E$ (twice the normalized kinetic energy) then reduces to the form

$$\frac{dE}{d\theta}(kE - \cos\theta) = 2E(P - E - \sin\theta), \tag{54}$$

and we have an Abel equation of the second kind [2]. The well-known canonical transformations for this equation can be used to reduce (54) to the appropriate canonical form with one arbitrary function. As a result, we can now predict the solvable cases for Eq. (54) even when the control variables are not constants but are functions of the angle of inclination of the trajectory $P(\theta)$ and $k(\theta)$. Given the additional assumption $P = V^2$ in (53) (constant-energy maneuvering), Eq. (54) admits the additional symmetry operator

$$X_{10} = \frac{1}{V \sin\theta} \partial_\theta, \tag{55}$$

which can be used to construct a whole series of exact solutions (the details can be found in [3]).

4. CONCLUSION

The analysis of an example illustrates the high "resolution" of group analysis of the system (1), (2); only three of the nine symmetry operators found can be determined by means of the π-theorem. With the aid of the above-determined symmetry operators it can be confirmed that two essential parameters can be formed out of the eight parameters of the system (1), (2): the thrust–weight ratio (P) and the lift–drag ratio (k). The application of group methods leads to constructive algorithms for seeking the required transformations, which the investigator can use to find the substitutions needed for any problem at hand.

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