https://orcid.org/0000-0002-9143-9398
https://orcid.org/0000-0001-8505-2878

UDC 519.21

N.V. KRUGLOVA[*], O.O. DYKHOVYCHNYI[*]

ON THE DISTRIBUTION OF THE MAXIMUM OF A BROWNIAN SHEET RESTRICTED TO A LOWER-DIMENSIONAL SET

[*]National Technical University of Ukraine «Igor Sikorsky Kyiv Polytechnic Institute», Kyiv, Ukraine

Анотація. *У статті освітлені достатні умови для знаходження розподілу функціоналів від* n*-вимірного броунівського листа (поля Ченцова) на множинах розмірності меншої, ніж розмірність самого поля. Ці результати отримані за допомогою узагальнення відомої теореми Дуба про ідентичність трансформації деякого гауссівського і вінерівського процесів на випадок випадкових полів. Відомо, що розподіл функціоналів від броунівського листа на множинах меншої розмірності, ніж розмірність самого поля, зокрема, таких як супремум на ламаних, призводить до необхідності розгляду досить складних інтегралів, для яких навіть оцінка стає проблемою. В роботі запропонований інший підхід до вирішення цієї задачі. Ми розглядаємо ймовірність того, що супремум броунівського листа є меншим, ніж деяка функція зносу. Отримані результати можуть істотно спростити задачу знаходження розподілу функціоналів від броунівського листа, зводячи її до задачі знаходження розподілу на паралелепіпедах розмірності меншої, ніж розмірність поля. В роботі також запропоновано низку прикладів, які ілюструють справедливість отриманої теореми шляхом моделювання відповідних полів і порівняння емпіричних і теоретичних імовірностей. Для моделювання було використано мову статистичного програмування R. При моделюванні броунівського листа використано розроблений раніше спеціальний алгоритм, що дозволяє моделювати випадкові поля з коваріаційними функціями спеціального виду, а також їх звуження на множини меншої розмірності (криві, ламані). Наведено відповідні програмні коди, а також результати моделювання.*

Ключові слова: *вінерівський процес, поле Ченцова, броунівський лист, розподіл супремума броунівського листа, теорема Дуба про трансформацію.*

Abstract. *The paper describes sufficient conditions for determining the distribution of functionals of an n-dimensional Brownian sheet (Chentsov field) on a set with the dimension lower than the dimension of the field. The results have been obtained through the generalization of the well-known Doob's Theorem on the identity of the transformation of some Gaussian and Wiener processes in the case of random fields. It is known that the distribution of functionals of a Brownian sheet on the sets with the dimension lower than the dimension of the field and a set of suprema of a piecewise linear curve in particular requires dealing with rather cumbersome integrals that might be hard to estimate. The paper suggests an alternative approach to the problem and considers the probability of suprema of a Brownian sheet being less than a certain drift. The obtained results can significantly simplify the task of determining the distribution of functionals of a Brownian sheet by reducing it to the problem of finding distribution on parallelepipeds with the dimension lower than the dimension of the field. There is also illustrated the validity of the obtained theorem through modeling some Brownian sheets and comparing empirical probabilities with theoretical ones. For the simulations, there have been used the R statistical software. In order to model a Brownian sheet, there has been utilized a special algorithm allowing to model random fields with covariance function of a special form as well as their restrictions on sets of lower dimension (curves, piecewise linear curves). The corresponding R code is provided as well.*

Keywords: *Wiener process, Chentsov random field, Brownian sheet, distribution of the supremum of a Brownian sheet, Doob's Transformation Theorem.*

DOI: 10.34121/1028-9763-2021-3-71-80

1. Introduction

The paper considers the distribution of an n-dimensional Brownian sheet (Chentsov field) $X_n(\vec{s}) = X_n(s_1,...,s_n)$ on a set S of the dimension lower than the dimension of the field. The Brownian sheet was first described by Chentsov [1] and Yeh [2]. In Russian-language literature it was previously known as Chenstov field, whereas in English-language literature the name Wiener-Yeh field was commonly used. Now, it is usually referred to as a Brownian sheet. It is known that random functions such as Brownian sheet occur in modeling of external effects influencing a system at a random moment in time and at a random location. For instance, this is a common scenario in the problem of modeling small transverse vibrations of a string under the influence of random external forces or in the problem of heat transfer in a rod with the presence of random heating/cooling sources [3, 4] as well as in filtration problems [5].

The problem of finding distribution on a set of the dimension lower than the dimension of the field occurs, for instance, in percolation theory [6].

Although some research on the distribution of the maximum of a Brownian sheet on a unit square has been conducted, no exact results have been obtained yet.

For example, a lot of results regarding the distribution of superemum of a field restricted to curves and polylines have been obtained. For the case of 2-dimensional field on the unit square, Park, Paranjape [7–9] have obtained the distribution of superemum of the field restricted to a polyline with a single vertex. In addition, considering the limit of the polyline, they have obtained distribution on the boundary of a square. Generalization of these results for a polyline with n vertices was considered by Klesov and Kruglova [10–12]. However, the obtained exact distribution of supreremum is given in terms of quite cumbersome integrals direct estimation of which may become problematic. That is why Kruglova and Dykhovychnyi [13] suggested an empirical approach to finding the distribution by modeling corresponding restriction to polylines. The suggested method is based on Doob’s Transformation Theorem [14].

The problem of finding the following probability is of a special interest:

$$P\left\{\sup_{0\le t\le T} w(t) - f(t) < 0\right\},$$

where $w(t)$ is a Wiener process and $f(t)$ is a deterministic function (drift).

For a n-dimensional Brownian sheet $X_n(s_1,...,s_n)$, similar problem can be generalized as follows:

$$P\left\{\sup_S (X_n(s_1,...,s_n) - g(s_1,...,s_n)) < 0\right\}, \tag{1}$$

where S is a set of the dimension lower than n and $g(\cdot)$ is a continuous function.

A preprint of the paper was published on arXiv (preprint arXiv:2006.06243 (2020)) [15].

The aim of this article is to simplify the problem of determining the distribution of functionals of a Brownian sheet by reducing it to the problem of finding distribution on parallelepipeds with the dimension lower than the dimension of the field.

2. Definitions and preliminaries

Let us denote $\wedge$ and $\vee$ as minimum and maximum respectively.

Definition 2.1. A real-valued separable Gaussian stochastic process $\{X_n(\vec{s}), \vec{s} = (s_1,...,s_n) \in [0,1]^n\}$ is called a Brownian sheet with n parameters, if $X_n(\vec{s})$ is such that:

1. $X_n(\vec{s}) = 0$, if $s_1 \cdot \ldots \cdot s_n = 0$.
2. $E\left[X_n(\vec{s})\right] = 0$, for all $\vec{s} \in [0,1]^n$.
3. $E\left[X_n(\vec{s})X_n(\vec{t})\right] = \prod_{i=1}^n (s_i \wedge t_i)$, for all $\vec{s}, \vec{t} = (t_1, \ldots, t_n) \in [0,1]^n$.

Theorem 2.1. (Doob's Transformation Theorem [14]) Let $Y(t)$ be a Gaussian process with $E[Y(t)] = 0$, for all $t \geq 0$, and covariance function

$$R(s,t) = u(s)v(t), \quad s \leq t. \tag{2}$$

If the function $a(t) = u(t)/v(t)$ is continuous and strictly increasing with the inverse $a^{-1}(t)$, then $w(t)$ and $Y(a^{-1}(t))/v(a^{-1}(t))$ are identical processes.

3. Main results

3.1. Generalization of Doob's Transformation Theorem

In order to solve the main problem (1), the following analogue of Theorem 2.1 is necessary.

Lemma 3.1. Let $Y(\vec{s})$ be a Gaussian field with $\mathrm{E}\left[Y(\vec{s})\right] = 0$, for all $\vec{s} = (s_1, \ldots, s_n) \in [0, \infty)^n$, and covariance function

$$\mathrm{E}\left[Y(\vec{s})Y(\vec{t})\right] = \prod_{i=1}^{n} u_i(s_i \wedge t_i) v_i(s_i \vee t_i) \tag{3}$$

for all $\vec{s}, \vec{t} = (t_1, \ldots, t_n) \in [0, \infty)^n$.

If the functions $a_i(t) = \dfrac{u_i(t)}{v_i(t)}$, $i = \overline{1,n}$ are increasing and continuous, then the field $Z(\vec{t}) = \dfrac{Y(a_1^{-1}(t_1), \ldots, a_n^{-1}(t_n))}{v_1(a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(t_n))}$ and Brownian sheet $X_n(\vec{t})$ are identical fields, where $a_i^{-1}(\cdot)$, $i = \overline{1,n}$ denotes the inverse of $a_i(\cdot)$.

Proof.

Let us calculate the expectation and the covariance function of $Z(\vec{t})$. It is clear that $\mathrm{E}\left[Z(\vec{t})\right] = 0$. Then, using (3), we get

$$E\left[Z(\vec{s})Z(\vec{t})\right] = E\left[\frac{Y(a_1^{-1}(s_1), \ldots, a_n^{-1}(s_n))}{v_1(a_1^{-1}(s_1)) \cdot \ldots \cdot v_n(a_n^{-1}(s_n))} \frac{Y(a_1^{-1}(t_1), \ldots, a_n^{-1}(t_n))}{v_1(a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(t_n))}\right] =$$

$$= \frac{u_1(a_1^{-1}(s_1) \wedge a_1^{-1}(t_1)) \cdot \ldots \cdot u_n(a_n^{-1}(s_n) \wedge a_n^{-1}(t_n))}{v_1(a_1^{-1}(s_1)) \cdot \ldots \cdot v_n(a_n^{-1}(s_n))} \times$$

$$\times \frac{v_1(a_1^{-1}(s_1) \vee a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(s_n) \vee a_n^{-1}(t_n))}{v_1(a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(t_n))} =$$

$$= \frac{u_1(a_1^{-1}(s_1) \wedge a_1^{-1}(t_1)) \cdot \ldots \cdot u_n(a_n^{-1}(s_n) \wedge a_n^{-1}(t_n))}{v_1(a_1^{-1}(s_1) \wedge a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(s_n) \wedge a_n^{-1}(t_n))} \times$$

$$\times \frac{v_1(a_1^{-1}(s_1) \vee a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(s_n) \vee a_n^{-1}(t_n))}{v_1(a_1^{-1}(s_1) \vee a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(s_n) \vee a_n^{-1}(t_n))} =$$

$$= a_1(a_1^{-1}(s_1) \wedge a_1^{-1}(t_1)) \cdot \ldots \cdot a_n(a_n^{-1}(s_n) \wedge a_n^{-1}(t_n)) =$$

$$= a_1(a_1^{-1}(s_1 \wedge t_1)) \cdot \ldots \cdot a_n(a_n^{-1}(s_n \wedge t_n)) = (s_1 \wedge t_1) \cdot \ldots \cdot (s_n \wedge t_n).$$

The penultimate equality holds because the functions $a_i^{-1}(\cdot)$, $i = \overline{1, n}$ are strictly increasing. Thus,

$$a_i^{-1}(s_i) \wedge a_i^{-1}(t_i) = a_i^{-1}(s_i \wedge t_i), \quad i = \overline{1, n}.$$

Using the definition of a Brownian sheet, we get

$$\mathrm{E}\left[X_n(\vec{s}) X_n(\vec{t})\right] = (s_1 \wedge t_1) \cdot \ldots \cdot (s_n \wedge t_n),$$

$$\mathrm{E}\left[X_n(\vec{t})\right] = 0.$$

Due to the fact that the Brownian sheet $X_n(\vec{t})$ and $Z(\vec{t})$ are both Gaussian, the field $Z(\vec{t})$ and $X_n(\vec{t})$ are identical.

Lemma 3.2. Let $Y(\vec{s})$ be a Gaussian field with $\mathrm{E}\left[Y(\vec{s})\right] = 0$ for all $\vec{s} = (s_1, \ldots, s_n) \in [0, \infty)^n$ and covariance function

$$\mathrm{E}\left[Y(\vec{s}) Y(\vec{t})\right] = \prod_{i=1}^{n} u_i(s_i \wedge t_i) v_i(s_i \vee t_i) \tag{4}$$

for all $\vec{s}, \vec{t} = \left(t_1, \ldots, t_n\right) \in [0, \infty)^n$.

If the functions $a_i(t) = \dfrac{u_i(t)}{v_i(t)}$, $i = \overline{1, n}$ are decreasing and continuous, then the field $Z(\vec{t}) = \dfrac{Y(a_1^{-1}(t_1), \ldots, a_n^{-1}(t_n))}{v_1(a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(t_n))}$ and Brownian sheet $X_n(\vec{t})$ are identical fields, where $a_i^{-1}(\cdot)$, $i = \overline{1, n}$ denotes the inverse of $a_i(\cdot)$.

Proof.

Let us calculate the expectation and the covariance function of $Z(\vec{t})$. Analogously to Lemma 3.1, we get $\mathrm{E}\left[Z(\vec{t})\right] = 0$. Then

$$E\left[Z(\vec{s}) Z(\vec{t})\right] = E\left[\frac{Y(a_1^{-1}(s_1), \ldots, a_n^{-1}(s_n))}{u_1(a_1^{-1}(s_1)) \cdot \ldots \cdot u_n(a_n^{-1}(s_n))} \frac{Y(a_1^{-1}(t_1), \ldots, a_n^{-1}(t_n))}{u_1(a_1^{-1}(t_1)) \cdot \ldots \cdot u_n(a_n^{-1}(t_n))}\right] =$$

$$\times \frac{u_1(a_1^{-1}(s_1) \wedge a_1^{-1}(t_1)) \cdot \ldots \cdot u_n(a_n^{-1}(s_n) \wedge a_n^{-1}(t_n))}{u_1(a_1^{-1}(s_1) \wedge a_1^{-1}(t_1)) \cdot \ldots \cdot u_n(a_n^{-1}(s_n) \wedge a_n^{-1}(t_n))} =$$

$$= \frac{v_1(a_1^{-1}(s_1) \vee a_1^{-1}(t_1)) \cdot \ldots \cdot v_n(a_n^{-1}(s_n) \vee a_n^{-1}(t_n))}{u_1(a_1^{-1}(s_1) \vee a_1^{-1}(t_1)) \cdot \ldots \cdot u_n(a_n^{-1}(s_n) \vee a_n^{-1}(t_n))} \times$$

$$= a_1(a_1^{-1}(s_1) \vee a_1^{-1}(t_1)) \cdot \ldots \cdot a_n(a_n^{-1}(s_n) \vee a_n^{-1}(t_n)) =$$

$$= a_1(a_1^{-1}(s_1 \wedge t_1)) \cdot \ldots \cdot a_n(a_n^{-1}(s_n \wedge t_n)) = (s_1 \wedge t_1) \cdot \ldots \cdot (s_n \wedge t_n).$$

The above holds because the functions $a_i^{-1}(\cdot)$, $i = \overline{1,n}$ are strictly decreasing. Therefore,

$$a_i^{-1}(s_i) \vee a_i^{-1}(t_i) = a_i^{-1}(s_i \wedge t_i), \quad i = \overline{1,n}.$$

Since
$$\mathrm{E}\left[X_n(\vec{s})X_n(\vec{t})\right] = (s_1 \wedge t_1) \cdot \ldots \cdot (s_n \wedge t_n),$$

$$\mathrm{E}\left[X_n(\vec{t})\right] = 0.$$

Therefore, smilary to Lemma 3.1, the field $Z(\vec{t})$ and $X_n(\vec{t})$ are identical.

3.2. The main theorem

Let $G = [0, y_1] \times \ldots \times [0, y_n] \subset [0,1]^n$.

Supposing that $d < n$, let us consider a d-dimensional surface $S \subset [0,1]^n$ given by

$$\begin{cases} s_{d+1} = f_{d+1}(s_1, \ldots, s_d) = \prod_{i=1}^{d} \varphi_{i,d+1}(s_i), \\ \ldots\ldots\ldots \\ s_n = f_n(s_1, \ldots, s_d) = \prod_{i=1}^{d} \varphi_{i,n}(s_i), \end{cases}$$

where $0 \le s_i \le y_i$, $i = \overline{1,n}$, $\varphi_{k,j}(s_i), k = \overline{1,d}, j = \overline{d+1,n}$ are decreasing functions.

Let us denote decreasing functions $z_i(\cdot), i = \overline{1,d}$, so that

$$\prod_{d+1}^{n} f_i(s_1, \ldots, s_d) = \prod_{i=1}^{d} z_i(s_i).$$

Let $a_i(t) = \dfrac{t}{z_i(t)}$ and $x_i = a_i(y_i)$, $i = \overline{1,d}$, and $D = I_1 \times \ldots \times I_d$, where $I_i = [0, x_i]$.

Theorem 3.1. Let $X_n(\vec{s})$ be an n-parameter Brownian sheet on G. Let $g_S(s_1,\ldots,s_d)$ denote the restriction of $g(\vec{s})$ to S. If for all $i = \overline{1,d}$, there exists $a_i^{-1} : I_i \to [0, y_i]$, the inverse of $a_i(\cdot)$, then

$$P\left\{\sup_S (X_n(\vec{s}) - g(\vec{s})) < 0\right\} =$$

$$= P\left\{\sup_D \left(X_d(t_1,\ldots,t_d) - \frac{g_S(a_1^{-1}(t_1),\ldots,a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1)) \cdot \ldots \cdot z_d(a_d^{-1}(t_d))} \right) < 0\right\},$$

where $X_d(t_1,...,t_d)$ is an d-parameter Brownian sheet on D.

Proof.

Let $X_S(s_1,\ldots,s_d)$ denote the restriction of $X(\vec{s})$ to S. The expectation and the covariance function of the field $X_S(s_1,\ldots,s_d)$ are

$$\mathrm{E}[X_S(s_1,\ldots,s_d)] = \mathrm{E}[X(s_1,\ldots,s_d, f_{d+1}(s_1,\ldots,s_d),\ldots, f_n(s_1,\ldots,s_d))] = 0.$$

Then $$\mathrm{E}[X_S(s_1,\ldots,s_d)X_S(t_1,\ldots,t_d)] =$$

$$= E[X(s_1,\ldots,s_d, f_{d+1}(s_1,\ldots,s_d),\ldots, f_n(s_1,\ldots,s_d))X(t_1,\ldots,t_d, f_{d+1}(t_1,\ldots,t_d),\ldots, f_n(t_1,\ldots,t_d))] =$$

$$= \prod_{i=1}^{d}(s_i \wedge t_i)\cdot \prod_{j=d+1}^{n}(f_j(s_1,\ldots,s_d)\wedge f_j(t_1,\ldots,t_d)) =$$

$$= \prod_{i=1}^{d}(s_i \wedge t_i)\cdot \prod_{i=1}^{d}\prod_{j=d+1}^{n}\varphi_{i,j}(s_i \vee t_i) = \prod_{i=1}^{d}(s_i \wedge t_i) z_i(s_i \vee t_i).$$

Since $z_i(t), i=\overline{1,d}$ are decreasing, $\dfrac{t}{z_i(t)}$, $i=\overline{1,d}$ are increasing. Hence, the assumptions of the theorem allow us to apply Lemma 3.1.

$$P\left\{\sup_S (X_n(\vec{s}) - g(\vec{s})) < 0\right\} =$$

$$= P\left\{\sup_{[0,y_1]\times\ldots\times[0,y_d]} (X_S(s_1,...,s_d) - g_S(s_1,...,s_d)) < 0\right\} =$$

$$= P\left\{\sup_D\left(\frac{X_S(a_1^{-1}(t_1),...,a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1))\cdot\ldots\cdot z_d(a_d^{-1}(t_d))} - \frac{g_S(a_1^{-1}(t_1),...,a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1))\cdot\ldots\cdot z_d(a_d^{-1}(t_d))}\right) < 0\right\} =$$

$$= P\left\{\sup_D\left(X_d(t_1,...,t_d) - \frac{g_S(a_1^{-1}(t_1),...,a_d^{-1}(t_d))}{z_1(a_1^{-1}(t_1))\cdot\ldots\cdot z_d(a_d^{-1}(t_d))}\right) < 0\right\}.$$

Therefore, using Theorem 3.1, it is possible to reduce the problem of finding the distribution of the maximum of $X_d(t_1,...,t_d)$ on S to the problem of finding its distribution on the parallelepiped D, which may be easier to deal with. For instance, this may be the case if one wants to study the asymptotic behavior of this distribution.

3.3. Examples

Let us consider the following two applications of Theorem 3.1.

Example 3.1. Let $X_3(s_1,s_2,s_3)$ be a 3-dimensional Brownian sheet ($n=3$ and $d=1$) and let S be a curve given by

$$\begin{cases} s_2 = \sqrt{1-s_1}, \\ s_3 = \sqrt{1-s_1}. \end{cases},$$

Let us suppose $g(s_1, s_2, s_3) = s_1 + s_2 s_3$. Then it can be shown that

$$P\left\{\sup_S (X(s_1, s_2, s_3) - g(s_1, s_2, s_3)) < 0\right\} = 1 - e^{-2}.$$

Solution. From the definition of S we have

$$z_1(s_1) = 1 - s_1,$$

$$a_1(s_1) = \frac{s_1}{1 - s_1}.$$

The inverse of $a_1(\cdot)$ is

$$a_1^{-1}(s_1) = \frac{s_1}{1 + s_1}.$$

Then
$$z_1(a_1^{-1}(s_1)) = \frac{1}{1 + s_1}$$

and
$$g_S(s_1) = s_1 + 1 - s_1 = 1.$$

Hence,
$$\frac{g_S(a_1^{-1}(s_1))}{z_1(a_1^{-1}(s_1))} = 1 + s_1.$$

$$P\left\{\sup_S (X_3(s_1, s_2, s_3) - s_1 - s_2 s_3) < 0\right\} =$$

$$= P\left\{\sup_{t \in [0,\infty)} (w(t) - 1 - t) < 0\right\} = 1 - e^{-2}.$$

The given above identity is due to the following result [14]:

$$P\left\{\sup_{0 \le t < \infty} (w(t) - at - b) < 0\right\} = 1 - e^{-2ab}. \tag{5}$$

Example 3.2. Let us suppose that $g(s_1, s_2, s_3) = s_1 + s_2^2 + s_3^2$ and S is the same as in Example 3.1. Then it can be shown that

$$P\left\{\sup_S (X_3(s_1, s_2, s_3) - g(s_1, s_2, s_3)) < 0\right\} = 1 - e^{-4}.$$

Solution. Since $g(s_1, s_2, s_3) = s_1 + s_2^2 + s_3^2$, we have

$$g_S(s_1) = s_1 + 1 - s_1 + 1 - s_1 = 2 - s_1,$$

$$\frac{g_S(a_1^{-1}(s_1))}{z_1(a_1^{-1}(s_1))} = 2 + s_1.$$

Finally, using Theorem 3.1, we get

$$P\left\{\sup_{S}(X_3(s_1,s_2,s_3)-s_1-s_2^2-s_3^2)<0\right\}=$$

$$=P\left\{\sup_{t\in[0,\infty)}(w(t)-2-t)<0\right\}=1-e^{-4}.$$

It is reasonable to compare the obtained probabilities with the corresponding empirical estimates resulting from simulation of the 3-dimensional Brownian sheet. Let us use the algorithm for the simulation of a Gaussian processes with special covariance function suggested in [13].

Below there is given an R code that simulates the Gaussian processes provided in Examples 3.1 and 3.2 and computes both empirical and theoretical distributions of their maximums.

```
> m<-numeric(10^4)
> n<-numeric(10^4)
> for(i in 1:10^4){
+   t<-seq(0,1-1/1000,length.out=1000)
+   vt<-1-t
+   at<-t/(1-t)
+   D<-diff(at)
+   y<-vt*c(0,cumsum(rnorm(999,0,sqrt(D))))  # the field
+   m[i]<-max(y)                    # compute maximum (Example 3.1)
+   n[i]<-max(y-2+t)                # compute maximum (Example 3.2)
+ }
> m1<-m[m<1]
> length(m1)/10^4
[1] 0.8683                        # empirical probability (Example 3.1)
> 1-exp(-2)
[1] 0.8646647                     # theoretical probability (Example 3.1)
> n1<-n[n<0]
> length(n1)/10^4
[1] 0.9833                        # empirical probability (Example 3.2)
> 1-exp(-4)
[1] 0.9816844                     # theoretical probability (Example 3.2)
```

It can be seen that in both cases the theoretical results are quite close to the empirical ones (increasing the number of points in the mesh that will clearly increase accuracy of the estimate).

Let us consider another example for a 4-dimensional Brownian sheet.

Example 3.3. Let $X_4(s_1,s_2,s_3,s_4)$ be a 4-parameter Brownian sheet. For $\lambda>0$, let us consider the following probability:

$$P_S(X)=P\left\{\sup_{(s_1,s_2,s_3,s_4)\in S}\left(X_4(s_1,s_2,s_3,s_4)-\frac{\lambda s_3 s_4}{(s_1+s_3)(s_2+s_4)}\right)<0\right\},\tag{6}$$

where $S=\left\{(s_1,s_2,s_3,s_4):0\le s_i\le\frac{1}{2}, i=\overline{1,4}, s_3=1-s_1, s_4=1-s_2\right\}$.

Solution. Let X_S denote the restriction of the field $X_4(s_1,s_2,s_3,s_4)$ to S. It is clear that $X_S(s_1,s_2)=X_4(s_1,s_2,1-s_1,1-s_2)$ and

$$R(\bar{s},\bar{t}) = E[X_S(\bar{s})X_S(\bar{t})] = \prod_{i=1}^{2}(s_i \wedge t_i)(1-s_i \vee t_i).$$

Using the notation of Theorem 3.1, we have

$$z_1(s_1) = 1-s_1, z_2(s_2) = 1-s_2.$$

Therefore, $$a_1(s_1) = \frac{s_1}{1-s_1}, a_2(s_2) = \frac{s_2}{1-s_2}$$

and $$a_1^{-1}(s_1) = \frac{s_1}{1+s_1}, a_2^{-1}(s_2) = \frac{s_2}{1+s_2}, y_1 = \frac{1}{2}, y_2 = \frac{1}{2}, x_1 = a_1(y_1) = 1, x_2 = a_1(y_1) = 1.$$

Using Theorem 3.1, we can rewrite the probability (5) in the following form:

$$P_S(X) = P\left\{ \sup_{(s_1,s_2)\in[0,1/2]^2} (X_S(s_1,s_2) - \lambda(1-s_1)(1-s_2)) < 0 \right\} =$$

$$= P\left\{ \sup_{(s_1,s_2)\in(0,1]^2} (1+s_1)(1+s_2)X_S\left(\frac{s_1}{1+s_1}, \frac{s_2}{1+s_2}\right) < \lambda \right\} =$$

$$= P\left\{ \sup_{(s_1,s_2)\in(0,1]^2} X_2(s_1,s_2) < \lambda \right\},$$

where $X(s_1,s_2)$ is a 2-dimensional Brownian sheet. In this way, the initial problem is reduced to the problem of finding the distribution of supremum of the Brownian sheet on a square.

4. Conclusion

In this paper there has been obtained a generalization of Doob's Transformation Theorem for a multidimensional Gaussian random field. This generalization has been used to reduce the problem of finding the distribution of supremum of a n-dimensional Brownian sheet restricted to a lower-dimensional set to the problem of finding its distribution on simpler sets (parallelepipeds).

With the help of a special algorithm it became possible to conduct modeling of Brownian sheet restrictions to various curves and obtain in the result their empirical distribution which appeared to coincide with the theoretical ones.

REFERENCES

1. Chentsov N.N. Wiener random fields depending on several parameters. *Doklady AN SSSR*. 1956. Vol. 46, N 4. P. 607–609 (in Russian).

2. Yeh J. Wiener Measure in a Space of Functions of Two Variables. *Transactions of the American Mathematical Society.* 1960. N 95. P. 433–450.

3. Allen E.J. *Modeling with Ito Stochastic Differential Equations*. Dordrecht: Springer, 2007. 228 p. doi: 10.1007/978-1-4020-5953-7.

4. Bovkun V.A. On models that lead to an infinite-dimensional stochastic Cauchy problem. *Teor. Veroyatnost. i Primenen.* 2017. Vol. 62, N 4. P. 803–804 (in Russian). doi: 10.4213/tvp5151.

5. Davar Khoshnevisan. *Five Lectures on Brownian Sheet*. 2001. URL: http://www.math.utah.edu/?davar.

6. Men'shikov M.V., Molchanov S.A., Sidorenko A.F. Percolation theory and some applications. *J Math*

Sci. 1988. N 42. P. 1766–1810. URL: https://doi.org/10.1007/BF01095508.
7. Paranjape S. R., Park C. Distribution of the supremum of the two-parameter Yeh-Wiener process on the boundary. *J. Appl. Probab*. 1973. Vol. 10, N 4. P. 875–880.
8. Park C., Schuurmann F. J. Evaluations of barrier-crossing probabilities of Wiener paths. *J. Appl. Prob.* 1976. N 13. P. 267–275.
9. Park C. Representations of Gaussian processes by Wiener processes. *Pacific Journal of Mathematics*. 1981. Vol. 94, N 2. P. 407–415.
10. Kruglova N.V. Distribution of the maximum of the Chentsov random field. *Theor. Stoch. Process.* 2008. N 1. P. 76–81.
11. Klesov O.I., Kruglova N.V. Distribution of the functionals of the Chentsov R3 random field. *Naukovi visti NTUU-KPI*. 2007. N 6. P. 145–150 (in Ukrainian).
12. Prokhorenko N.V. Stochastic Equivalence of Gaussian Process to the Wiener Process, Brownian Bridge, Ornstein-Uhlenbeck Process. *Naukovi visti NTUU-KPI.* 2016. N 4. P. 85–93 (in Ukrainian).
13. Dykhovychnyi O.O., Kruglova N.V., Virstiuk O.I. Methods for identification of probability distribution of random variables from data samples with R statistical computing language. *Mathematics in Modern Technical University.* 2018. N 1. P. 91–100 (in Ukrainian).
14. Doob J.L. Heuristic approach to Kolmogorov-Smirnov theorems. *Ann. Math. Statist.* 1949. N 20. P. 393–403.
15. Kruglova N., Dykhovychnyi O. On the distribution of the maximum of a Brownian sheet restricted to a lower-dimensional set. Preprint arXiv: 2006.06243. 2020. 8 p.

Стаття надійшла до редакції 15.06.2021